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TRANSPOSITION ANTI-INVOLUTION IN CLIFFORD ALGEBRAS AND INVARIANCE GROUPS OF SCALAR PRODUCTS ON SPINOR SPACES

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Abstract. *We introduce on the abstract level in real Clifford algebras $C\ell_{p,q}$ of a non-degenerate quadratic space (V,Q), where Q has signature $\varepsilon = (p,q)$, a transposition anti-involution $T_\varepsilon^{\tilde{}}$. In a spinor representation, the anti-involution $T_\varepsilon^{\tilde{}}$ gives transposition, complex Hermitian conjugation or quaternionic Hermitian conjugation when the spinor space $\check{S}$ is viewed as a $C\ell_{p,q}$-left and $\check{\mathbb{K}}$-right module with $\check{\mathbb{K}}$ isomorphic to $\mathbb{R}$ or ${}^2\mathbb{R}$, $\mathbb{C}$, or, $\mathbb{H}$ or ${}^2\mathbb{H}$. This map and its application to SVD was first presented at ICCA 7 in Toulouse in 2005 [3].*

The anti-involution $T_\varepsilon^{\tilde{}}$ is a lifting to $C\ell_{p,q}$ of an orthogonal involution $t_\varepsilon : V \to V$ which depends on the signature of Q. The involution is a symmetric correlation [18] $t_\varepsilon : V \to V^* \cong V$ and it allows one to define a reciprocal basis for the dual space (V^*, Q). When the Clifford algebra $C\ell_{p,q}$ splits into the graded tensor product $C\ell_{p,0} \hat{\otimes} C\ell_{0,q}$, the anti-involution $T_\varepsilon^{\tilde{}}$ acts as reversion on $C\ell_{p,0}$ and as conjugation on $C\ell_{0,q}$. Using the concept of a transpose of a linear mapping one can show that if $[L_u]$ is a matrix in the left regular representation of the operator $L_u : C\ell_{p,q} \to C\ell_{p,q}$ relative to a Grassmann basis $\mathcal{B}$ in $C\ell_{p,q}$, then matrix $[L_{T_\varepsilon^{\tilde{}}(u)}]$ is the matrix transpose of $[L_u]$, see [6].

Of particular importance is the action of $T_\varepsilon^{\tilde{}}$ on the spinor space. The algebraic spinor space $\check{S}$ is realized as a left minimal ideal generated by a primitive idempotent f, or a sum $f + \hat{f}$ in simple or semisimple algebras as in [14]. The map $T_\varepsilon^{\tilde{}}$ allows us to define a new spinor scalar product $\check{S} \times \check{S} \to \check{\mathbb{K}}$, where $\mathbb{K} = fC\ell_{p,q}f$ and $\check{\mathbb{K}} = \mathbb{K}$ or $\mathbb{K} \oplus \hat{\mathbb{K}}$ depending whether the algebra is simple or semisimple. Our scalar product is in general different from the two scalar products discussed in literature, e.g., [14]. However, it reduces to one or the other in Euclidean and anti-Euclidean signatures. The anti-involution $T_\varepsilon^{\tilde{}}$ acts as the identity map, complex conjugation, or quaternionic conjugation on $\check{\mathbb{K}}$. Thus, the action of $T_\varepsilon^{\tilde{}}$ on spinors results in matrix transposition, complex Hermitian conjugation, or quaternionic Hermitian conjugation. We classify automorphism group of the new product as $O(N)$, $U(N)$, $Sp(N)$, ${}^2O(N)$, or ${}^2Sp(N)$.

1 INTRODUCTION

Let $C\ell_n$ be a universal Clifford algebra over an n-dimensional real quadratic space (V, Q) with $Q(\mathbf{x}) = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \cdots + \varepsilon_n x_n^2$ where $\varepsilon_i = \pm 1$ and $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \in V$ for an orthonormal basis $\mathcal{B}_1 = \{\mathbf{e}_i\}_{i=1}^n$. Let $\mathcal{B}$ be the canonical basis of $\bigwedge V$ generated by $\mathcal{B}_1$. That is, let $[n] = \{1, 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$, by underlined Roman characters. The basis elements of $\bigwedge V$, or, of $C\ell_n$ due to the linear space isomorphism $\bigwedge V \to C\ell_n$ [14], can be indexed by these finite ordered subsets as $\mathbf{e}_{\underline{i}} = \wedge_{i\in\underline{i}}\, \mathbf{e}_i$. Then, an arbitrary element of $\bigwedge V \cong C\ell_n$ can be written as $u = \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\mathbf{e}_{\underline{i}}$ where $u_{\underline{i}} \in \mathbb{R}$ for each $\underline{i} \in 2^{[n]}$. The unit element 1 of $C\ell_n$ is identified with $\mathbf{e}_\emptyset$. Our preferred basis for $C\ell_n$ is the exterior algebra basis $\mathcal{B}$ sorted by an *admissible* monomial order $\prec$ on $\bigwedge V$. We choose for $\prec$ the monomial order called `InvLex`, or, the *inverse lexicographic order* [4, 5]. Let B be the symmetric bilinear form defined by Q and let $<\cdot,\cdot> : \bigwedge V \times \bigwedge V \to \mathbb{R}$ be an extension of B to $\bigwedge V$ [14]. We will need this extension later when we define the Clifford algebra $C\ell(V^*, Q)$.

We begin by defining the following map on (V, Q) dependent on the signature ε of Q.

Definition 1. Let $t_\varepsilon : V \to V$ be the linear map defined as

$$t_\varepsilon(\mathbf{x}) = t_\varepsilon(\sum_{i=1}^{n} x_i\mathbf{e}_i) = \sum_{i=1}^{n} x_i \left(\frac{\mathbf{e}_i}{\varepsilon_i}\right) = \sum_{i=1}^{n} x_i\,(\varepsilon_i\mathbf{e}_i) \tag{1}$$

for any $\mathbf{x} \in V$ and for the orthonormal basis $\mathcal{B}_1 = \{\mathbf{e}_i\}_{i=1}^n$ in V diagonalizing Q.

The t_ε map can be viewed in two ways: (1) As a linear orthogonal involution of V; (2) As a *correlation* [18] mapping $t_\varepsilon : V \to V^* \cong V$. The set of vectors $\mathcal{B}_1^* = \{t_\varepsilon(\mathbf{e}_i)\}_{i=1}^n$ gives an orthonormal basis in the dual space (V^*, Q). Furthermore, under the identification $V \cong V^*$, t_ε is a symmetric non-degenerate correlation on V thus making the pair (V, t_ε) into a *non-degenerate real correlated (linear) space* [6]. Then, viewing t_ε as a correlation $V \to V^*$, we can define the action of $t_\varepsilon(\mathbf{x}) \in V^*$ on $\mathbf{y} \in V$ for any $\mathbf{x} \in V$ as

$$t_\varepsilon(\mathbf{x})(\mathbf{y}) = <t_\varepsilon(\mathbf{x}), \mathbf{y}>, \tag{2}$$

and we get the expected duality relation among the basis elements in $\mathcal{B}_1$ and $\mathcal{B}_1^*$:

$$t_\varepsilon(\mathbf{e}_i)(\mathbf{e}_j) = <\varepsilon_i\mathbf{e}_i, \mathbf{e}_j> = \varepsilon_i<\mathbf{e}_i, \mathbf{e}_j> = \varepsilon_i\varepsilon_j\delta_{i,j} = \delta_{i,j}. \tag{3}$$

The extension of the duality $V \to V^*$ to the Clifford algebras $C\ell(V, Q) \to C\ell(V^*, Q)$ is of fundamental importance to defining a new transposition scalar product on spinor spaces. When we apply Porteous' theorem [18, Thm. 15.32] to the involution t_ε, we get the following theorem and its corollary proven in [6].[1]

Proposition 1. *Let $\mathcal{A} = C\ell_n$ be the universal Clifford algebra of (V, Q) and let $t_\varepsilon : V \to V$ be the orthogonal involution of V defined in (1). Then there exists a unique algebra involution T_ε of $\mathcal{A}$ and a unique algebra anti-involution $T_\varepsilon^{\,\tilde{}}$ of $\mathcal{A}$ such that the following diagrams commute:*

$$\begin{tikzcd} V \arrow[r, "t_\varepsilon"] \arrow[d, tail, "\iota"] & V \arrow[d, tail, "\iota"] \\ \mathcal{A} \arrow[r, "T_\varepsilon"] & \mathcal{A} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} V \arrow[r, "t_\varepsilon"] \arrow[d, tail, "\iota"] & V \arrow[d, tail, "\iota"] \\ \mathcal{A} \arrow[r, "T_\varepsilon^{\,\tilde{}}"] & \mathcal{A} \end{tikzcd} \tag{4}$$

In particular, we can define T_ε and $T_\varepsilon^{\,\tilde{}}$ as follows:

[1] We view $C\ell_n$ as Porteous' $\mathbf{L}^\alpha$*-Clifford algebra for* (V, Q) under the identification $\mathbf{L} = \mathbb{R}$ and $\alpha = 1_{\mathbb{R}}$.

(i) For simple k-vectors $\mathbf{e}_{\underline{i}}$ in $\mathcal{B}$, let $T_\varepsilon(\mathbf{e}_{\underline{i}}) = T_\varepsilon(\prod_{i\in\underline{i}} \mathbf{e}_i) = \prod_{i\in\underline{i}} t_\varepsilon(\mathbf{e}_i)$ where $k = |\underline{i}|$ and $T_\varepsilon(1_{\mathcal{A}}) = 1_{\mathcal{A}}$. Then, extend by linearity to all of $\mathcal{A}$.

(ii) For simple k-vectors $\mathbf{e}_{\underline{i}}$ in $\mathcal{B}$, let

$$T_\varepsilon\tilde{}(\mathbf{e}_{\underline{i}}) = T_\varepsilon\tilde{}(\prod_{i\in\underline{i}} \mathbf{e}_i) = (\prod_{i\in\underline{i}} t_\varepsilon(\mathbf{e}_i))\tilde{} = (-1)^{\frac{k(k-1)}{2}} \prod_{i\in\underline{i}} t_\varepsilon(\mathbf{e}_i) \tag{5}$$

where $k = |\underline{i}|$ and $T_\varepsilon\tilde{}(1_{\mathcal{A}}) = 1_{\mathcal{A}}$. Then, extend by linearity to all of $\mathcal{A}$.

Maple code of the procedure `tp` which implements the anti-involution $T_\varepsilon\tilde{}$ in $C\ell_n$, was first presented at ICCA 7 in Toulouse [3]. The procedure `tp` requires the `CLIFFORD` package [9]. In the following corollary, α, β, γ denote, respectively, the grade involution, the reversion, and the conjugation in $C\ell_n$.

Corollary 1. *Let $\mathcal{A} = C\ell_{p,q}$ and let $T_\varepsilon : \mathcal{A} \to \mathcal{A}$ and $T_\varepsilon\tilde{} : \mathcal{A} \to \mathcal{A}$ be the involution and the anti-involution of $\mathcal{A}$ from Proposition 1.*

(i) For the Euclidean signature $(p, q) = (n, 0)$, or $p - q = n$, we have $t_\varepsilon = 1_V$. Thus, T_ε is the identity map $1_{\mathcal{A}}$ on $\mathcal{A}$ and $T_\varepsilon\tilde{}$ is the reversion β of $\mathcal{A}$.

(ii) For the anti-Euclidean signature $(p, q) = (0, n)$, or $p - q = -n$, we have $t_\varepsilon = -1_V$. Thus, T_ε is the grade involution α of $\mathcal{A}$ and $T_\varepsilon\tilde{}$ is the conjugation γ of $\mathcal{A}$.

(iii) For all other signatures $-n < p - q < n$, we have $t_\varepsilon = 1_{V_1} \otimes -1_{V_2}$ where $(V, Q) = (V_1, Q_1) \perp (V_2, Q_2)$. Here, (V_1, Q_1) is the Euclidean subspace of (V, Q) of dimension p spanned by $\{\mathbf{e}_i\}_{i=1}^p$ with $Q_1 = Q|_{V_1}$ while (V_2, Q_2) is the anti-Euclidean subspace of (V, Q) of dimension q spanned by $\{\mathbf{e}_i\}_{i=p+1}^{n=p+q}$ with $Q_2 = Q|_{V_2}$. Let $\mathcal{A}_1 = C\ell(V_1, Q_1)$ and $\mathcal{A}_2 = C\ell(V_2, Q_2)$ so $C\ell(V, Q) \cong C\ell(V_1, Q_1) \hat{\otimes} C\ell(V_2, Q_2)$. Let S (resp. $\hat{S}$) be the ungraded switch *(resp. the* graded switch*) on $C\ell(V_1, Q_1) \hat{\otimes} C\ell(V_2, Q_2)$.*[2] *Then,*

$$T_\varepsilon = 1_{\mathcal{A}_1} \otimes \alpha_{\mathcal{A}_2} \quad \text{and} \quad T_\varepsilon\tilde{} = (\beta_{\mathcal{A}_1} \otimes \gamma_{\mathcal{A}_2}) \circ (\hat{S} \circ S).$$

(iv) The anti-involution $T_\varepsilon\tilde{}$ is related to the involution T_ε through the reversion β as follows: $T_\varepsilon\tilde{} = T_\varepsilon \circ \beta = \beta \circ T_\varepsilon$.

For an extensive discussion of the properties of the involutions $T_\varepsilon\tilde{}$ and T_ε see [6].

Since (V^*, Q) is a non-degenerate quadratic space spanned by the orthonormal basis $\mathcal{B}_1^*$, we can define the Clifford algebra $C\ell(V^*, Q)$ as expected.

Definition 2. The *Clifford algebra over the dual space V^** is the universal Clifford algebra $C\ell(V^*, Q)$ of the quadratic pair (V^*, Q). For short, we denote this algebra by $C\ell_n^*$.[3]

[2] The switches are defined on the basis tensors $\mathbf{e}_{\underline{i}} \hat{\otimes} \mathbf{e}_{\underline{j}} \in C\ell_{p,0} \hat{\otimes} C\ell_{0,q}$ as $S(\mathbf{e}_{\underline{i}} \hat{\otimes} \mathbf{e}_{\underline{j}}) = \mathbf{e}_{\underline{j}} \hat{\otimes} \mathbf{e}_{\underline{i}}$ and $\hat{S}(\mathbf{e}_{\underline{i}} \hat{\otimes} \mathbf{e}_{\underline{j}}) = (-1)^{|\underline{i}||\underline{j}|} \mathbf{e}_{\underline{j}} \hat{\otimes} \mathbf{e}_{\underline{i}}$. Then, their action is extended by linearity to the graded product $C\ell_{p,0} \hat{\otimes} C\ell_{0,q}$ [6]

[3] Although from now on we denote the Clifford algebra of the dual (V^*, Q) via $C\ell_n^*$, we do not claim that $C\ell_n^*$ is the *dual* algebra of $C\ell_n$ in categorical sense as it was considered in [16] and references therein.

Let $\mathcal{B}^*$ be the canonical basis of $\bigwedge V^* \cong C\ell_n^*$ generated by $\mathcal{B}_1^*$ and sorted by `InvLex`. That is, we define $\mathcal{B}^* = \{T_\varepsilon(\mathbf{e}_{\underline{i}}) \,|\, \mathbf{e}_{\underline{i}} \in \mathcal{B}\}$ given that

$$<T_\varepsilon(\mathbf{e}_{\underline{i}}), \mathbf{e}_{\underline{j}}> = \delta_{\underline{i},\underline{j}} \tag{6}$$

for $\mathbf{e}_{\underline{i}}, \mathbf{e}_{\underline{j}} \in \mathcal{B}$ and $T_\varepsilon(\mathbf{e}_{\underline{i}}) \in \mathcal{B}^*$. An arbitrary linear form φ in $\bigwedge V^* \cong C\ell_n^*$ can be written as

$$\varphi = \sum_{\underline{i} \in 2^{[n]}} \varphi_{\underline{i}} T_\varepsilon(\mathbf{e}_{\underline{i}}) \tag{7}$$

where $\varphi_{\underline{i}} \in \mathbb{R}$ for each $\underline{i} \in 2^{[n]}$. Due to the linear isomorphisms $V \cong V^*$ and $\bigwedge V^* \cong C\ell(V^*, Q)$, we extend, by a small abuse of notation, the inner product $<\cdot,\cdot>$ defined in $\bigwedge V$ to

$$<\cdot,\cdot> : \bigwedge V^* \times \bigwedge V^* \to \mathbb{R}. \tag{8}$$

In this way we find, as expected, that the matrix of this inner product on $\bigwedge V^*$ is also diagonal, that is, that the basis $\mathcal{B}^*$ is orthonormal with respect to $<\cdot,\cdot>$. We extend the action of dual vectors from V^* on V to all linear forms φ in $C\ell_n^*$ acting on multivectors v in $C\ell_n$ via the inner product (8) as

$$\varphi(v) = <\varphi, v> = \sum_{\underline{i} \in 2^{[n]}} \varphi_{\underline{i}} v_{\underline{i}} \tag{9}$$

given that $\varphi = \sum_{\underline{i} \in 2^{[n]}} \varphi_{\underline{i}} T_\varepsilon(\mathbf{e}_{\underline{i}}) \in C\ell_n^*$ where $\varphi_{\underline{i}} = \varphi(\mathbf{e}_{\underline{i}}) \in \mathbb{R}$ and $v = \sum_{\underline{i} \in 2^{[n]}} v_{\underline{i}} \mathbf{e}_{\underline{i}} \in C\ell_n$ for some coefficients $v_{\underline{i}} \in \mathbb{R}$.

Properties of the left multiplication operator $L_u : C\ell_n \to C\ell_n,\ v \mapsto uv,\ \forall v \in C\ell_n$ and its dual $L_{\tilde{u}}$ with respect to the inner product $<\cdot,\cdot> : \bigwedge V \times \bigwedge V \to \mathbb{R}$ are discussed in [6]. In particular, it is shown there that if $[L_u]$ is the matrix of the operator L_u relative to the basis $\mathcal{B}$ and $\left[L_{T_\varepsilon^{\,\tilde{}}(u)}\right]$ is the matrix of the operator $L_{T_\varepsilon^{\,\tilde{}}(u)}$ relative to the basis $\mathcal{B}$, then $[L_u]^T = \left[L_{T_\varepsilon^{\,\tilde{}}(u)}\right] = \left[L_{T_\varepsilon(\tilde{u})}\right]$ where $[L_u]^T$ is the matrix transpose of $[L_u]$. However, in order to introduce a new scalar product on spinor spaces related to the involution $T_\varepsilon^{\,\tilde{}}$, we need to discuss the action of $T_\varepsilon^{\,\tilde{}}$ on spinor spaces.

2 ACTION OF THE TRANSPOSITION INVOLUTION ON SPINOR SPACES

Stabilizer groups $G_{p,q}(f)$ of primitive idempotents f are classified in [7]. The stabilizer $G_{p,q}(f)$ is a normal subgroup of Salingaros' finite vee group $G_{p,q}$ [20–22] which acts via conjugation on $C\ell_{p,q}$. The importance of the stabilizers to the spinor representation theory lies in the fact that a *transversal*[4] of $G_{p,q}(f)$ in $G_{p,q}$ generates spinor bases in $S = C\ell_{p,q} f$ and $\hat{S} = C\ell_{p,q}\hat{f}$. In [7] it is also shown that depending on the signature $\varepsilon = (p,q)$, the real anti-involution $T_\varepsilon^{\,\tilde{}}$ is responsible for transposition, the Hermitian complex, or the Hermitian quaternionic conjugation of a matrix $[u]$ for any u in all Clifford algebras $C\ell_{p,q}$ with the spinor representation realized either in S (simple algebras) or in $\check{S} = S \oplus \hat{S}$ (semisimple algebras). This is because $T_\varepsilon^{\,\tilde{}}$ acts on $\mathbb{K} = fC\ell_{p,q}f$ and $\check{\mathbb{K}} = \mathbb{K} \oplus \hat{\mathbb{K}}$ as an anti-involution. Thus, $T_\varepsilon^{\,\tilde{}}$ allows us to define a dual spinor space S^* or $\check{S}^*$, a new spinor product, and a new spinor norm. The following results are proven in [7].

[4] Let K be a subgroup of a group G. A *transversal* ℓ of K in G is a subset of G consisting of exactly one element $\ell(bK)$ from every (left) coset bK, and with $\ell(K) = 1$ [19].

Proposition 2. *Let $\psi, \phi \in S = C\ell_{p,q}f$. Then, $T_\varepsilon^{\tilde{}}(\psi)\phi \in \mathbb{K}$. In particular, $T_\varepsilon^{\tilde{}}(\psi)\psi \in \mathbb{R}f \subset \mathbb{K}$.*

Thus, we can define an invariance group of the scalar product $S \times S \to \mathbb{K}$, $(\psi, \phi) \mapsto T_\varepsilon^{\tilde{}}(\psi)\phi$, as follows:

Definition 3. Let $G^\varepsilon_{p,q} = \{g \in C\ell_{p,q} \mid T_\varepsilon^{\tilde{}}(g)g = 1\}$.

We find that $G_{p,q}(f) \trianglelefteq G_{p,q} \leq G^\varepsilon_{p,q} < C\ell^\times_{p,q}$ (the group of units in $C\ell_{p,q}$). Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be a set of $N = 2^k$, $k = q - r_{q-p}$, mutually annihilating primitive idempotents adding up to 1 in a simple Clifford algebra $C\ell_{p,q}$.[5] The set $\mathcal{F}$ constitutes one orbit under the action of $G_{p,q}$ [7].

Proposition 3. *Let $C\ell_{p,q}$ be a simple Clifford algebra, $p - q \neq 1 \bmod 4$ and $p + q \leq 9$. Let $\psi_i \in S_i = C\ell_{p,q}f_i$, $f_i \in \mathcal{F}$, and let $[\psi_i]$ (resp. $[T_\varepsilon^{\tilde{}}(\psi_i)]$) be the matrix of ψ_i (resp. $T_\varepsilon^{\tilde{}}(\psi_i)$) in the spinor representation with respect to the ordered basis $\mathcal{S}_1 = [m_1 f_1, \ldots, m_N f_1]$ with $\alpha_i = m_i^2$.[6] Then,*

$$[T_\varepsilon^{\tilde{}}(\psi_i)] = \begin{cases} [\psi_i]^T & \text{if } p - q = 0, 1, 2 \bmod 8; \\ [\psi_i]^\dagger & \text{if } p - q = 3, 7 \bmod 8; \\ [\psi_i]^\ddagger & \text{if } p - q = 4, 5, 6 \bmod 8; \end{cases} \tag{10}$$

where T denotes transposition, $\dagger$ denotes Hermitian complex conjugation, and $\ddagger$ denotes Hermitian quaternionic conjugation.

This action of $T_\varepsilon^{\tilde{}}$ on $S = S_i$ extends to a similar action on $\hat{S}$, hence to $\check{S} = S \oplus \hat{S}$ as it is shown in [7, 8]. In particular, the product $(\psi, \phi) \mapsto T_\varepsilon^{\tilde{}}(\psi)\phi$ is invariant under two of the subgroups of $G^\varepsilon_{p,q}$: The Salingaros' vee group $G_{p,q} < G^\varepsilon_{p,q}$ and the stabilizer group $G_{p,q}(f)$ of a primitive idempotent f. Since the stabilizer group $G_{p,q}(f)$ and its subgroups play an important role in constructing and understanding spinor representation of Clifford algebras, we provide here a brief summary of related definitions and findings. See [8] for a complete discussion.

Primitive idempotents $f \in \mathcal{F} \subset C\ell_{p,q}$ formed out of commuting basis monomials $\mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k}$ in $\mathcal{B}$ with square 1 have the form $f = \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_1})\frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_2}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_k})$ where $k = q - r_{q-p}$. With any primitive idempotent f, we associate the following groups:
(i) The *stabilizer $G_{p,q}(f)$ of f* defined as

$$G_{p,q}(f) = \{m \in G_{p,q} \mid mfm^{-1} = f\} < G_{p,q}. \tag{11}$$

The stabilizer $G_{p,q}(f)$ is a normal subgroup of $G_{p,q}$. In particular,

$$|G_{p,q}(f)| = \begin{cases} 2^{1+p+r_{q-p}}, & p - q \neq 1 \bmod 4; \\ 2^{2+p+r_{q-p}}, & p - q = 1 \bmod 4. \end{cases} \tag{12}$$

(ii) An abelian *idempotent group $T_{p,q}(f)$ of f*, a subgroup of $G_{p,q}(f)$ defined as

$$T_{p,q}(f) = \langle \pm 1, \mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k} \rangle < G_{p,q}(f), \tag{13}$$

where $k = q - r_{q-p}$.

[5] Here, r_i is Radon-Hurwitz number defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0, r_1 = 1, r_2 = r_3 = 2, r_4 = r_5 = r_6 = r_7 = 3$ [13, 14].

[6] For the sake of consistency with a proof of this proposition given in [7] we remark that α_i is just the square of the monomial $m_i^2 \in \{\pm 1\}$.

(iii) A *field group* $K_{p,q}(f)$ *of* f, a subgroup of $G_{p,q}(f)$, related to the (skew double) field $\mathbb{K} \cong fC\ell_{p,q}f$, and defined as

$$K_{p,q}(f) = \langle \pm 1, m \mid m \in \mathcal{K} \rangle < G_{p,q}(f) \tag{14}$$

where $\mathcal{K}$ is a set of Grassmann monomials in $\mathcal{B}$ which provide a basis for $\mathbb{K} = fC\ell_{p,q}f$ as a real subalgebra of $C\ell_{p,q}$.

The following theorem proven in [8] relates the above groups to $G_{p,q}$ and its commutator subgroup $G'_{p,q}$.[7]

Theorem 1. *Let f be a primitive idempotent in a simple or semisimple Clifford algebra $C\ell_{p,q}$ and let $G_{p,q}$, $G_{p,q}(f)$, $T_{p,q}(f)$, $K_{p,q}(f)$, and $G'_{p,q}$ be the groups defined above. Furthermore, let $S = C\ell_{p,q}f$ and $\mathbb{K} = fC\ell_{p,q}f$.*

(i) *Elements of $T_{p,q}(f)$ and $K_{p,q}(f)$ commute.*

(ii) $T_{p,q}(f) \cap K_{p,q}(f) = G'_{p,q} = \{\pm 1\}$.

(iii) $G_{p,q}(f) = T_{p,q}(f)K_{p,q}(f) = K_{p,q}(f)T_{p,q}(f)$.

(iv) $|G_{p,q}(f)| = |T_{p,q}(f)K_{p,q}(f)| = \frac{1}{2}|T_{p,q}(f)||K_{p,q}(f)|$.

(v) $G_{p,q}(f) \lhd G_{p,q}$, $T_{p,q}(f) \lhd G_{p,q}$, *and* $K_{p,q}(f) \lhd G_{p,q}$. *In particular, $T_{p,q}(f)$ and $K_{p,q}(f)$ are normal subgroups of $G_{p,q}(f)$.*

(vi) $G_{p,q}(f)/K_{p,q}(f) \cong T_{p,q}(f)/G'_{p,q}$ *and* $G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/G'_{p,q}$.

(vii) $(G_{p,q}(f)/G'_{p,q})/(T_{p,q}(f)/G'_{p,q}) \cong G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/\{\pm 1\}$ *and the transversal of $T_{p,q}(f)$ in $G_{p,q}(f)$ spans $\mathbb{K}$ over $\mathbb{R}$ modulo f.*

(viii) *A transversal of $G_{p,q}(f)$ in $G_{p,q}$ spans S over $\mathbb{K}$ modulo f.*

(ix) $(G_{p,q}(f)/T_{p,q}(f)) \lhd (G_{p,q}/T_{p,q}(f))$ *and* $(G_{p,q}/T_{p,q}(f))/(G_{p,q}(f)/T_{p,q}(f)) \cong G_{p,q}/G_{p,q}(f)$ *and a transversal of $T_{p,q}(f)$ in $G_{p,q}$ spans S over $\mathbb{R}$ modulo f.*

(x) *The stabilizer $G_{p,q}(f) = \bigcap_{x \in T_{p,q}(f)} C_{G_{p,q}}(x) = C_{G_{p,q}}(T_{p,q}(f))$ where $C_{G_{p,q}}(x)$ is the centralizer of x in $G_{p,q}$ and $C_{G_{p,q}}(T_{p,q}(f))$ is the centralizer of $T_{p,q}(f)$ in $G_{p,q}$.*

Recall that in `CLIFFORD` [9] information about each Clifford algebra $C\ell_{p,q}$ for $p + q \le 9$ is stored in a built-in data file. This information can be retrieved in the form of a seven-element list with the command `clidata([p,q])`. For example, for $C\ell_{3,0}$ we find:

$$\texttt{data} = [complex, 2, simple, \frac{1}{2}\mathrm{Id} + \frac{1}{2}\mathbf{e}_1, [\mathrm{Id}, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}], [\mathrm{Id}, \mathbf{e}_{23}], [\mathrm{Id}, \mathbf{e}_2]] \tag{15}$$

where Id denotes the identity element of the algebra. In particular, from the above we find that: (i) $C\ell_{3,0}$ is a simple algebra isomorphic to $\mathrm{Mat}(2, \mathbb{C})$; ($\texttt{data}[1]$, $\texttt{data}[2]$, $\texttt{data}[3]$) (ii) The expression $\frac{1}{2}\mathrm{Id} + \frac{1}{2}\mathbf{e}_1$ ($\texttt{data}[4]$) is a primitive idempotent f which may be used to generate a spinor ideal $S = C\ell_{3,0}f$; (iii) The fifth entry $\texttt{data}[5]$ provides, modulo f, a real basis for S, that is, $S = \mathrm{span}_{\mathbb{R}}\{f, \mathbf{e}_2 f, \mathbf{e}_3 f, \mathbf{e}_{23} f\}$; (iv) The sixth entry $\texttt{data}[6]$ provides, modulo f, a real basis for $\mathbb{K} = fC\ell_{3,0}f \cong \mathbb{C}$, that is, $\mathbb{K} = \mathrm{span}_{\mathbb{R}}\{f, \mathbf{e}_{23} f\}$; and, (v) The seventh entry $\texttt{data}[7]$ provides, modulo f, a basis for S over $\mathbb{K}$, that is, $S = \mathrm{span}_{\mathbb{K}}\{f, \mathbf{e}_2 f\}$.[8]

The above theorem yields the following corollary:

[7] We have $G'_{p,q} = \{1, -1\}$ since any two monomials in $G_{p,q}$ either commute or anticommute.

[8] See [1, 2] how to use `CLIFFORD`.

Corollary 2. *Let* `data` *be the list of data returned by the procedure* `clidata` *in* `CLIFFORD`*. Then,* `data`$[5]$ *is a transversal of $T_{p,q}(f)$ in $G_{p,q}$;* `data`$[6]$ *is a transversal of $T_{p,q}(f)$ in $G_{p,q}(f)$; and* `data`$[7]$ *is a transversal of $G_{p,q}(f)$ in $G_{p,q}$. Therefore,* $|\mathtt{data}[5]| = |\mathtt{data}[6]||\mathtt{data}[7]|$*. This is equivalent to* $|\frac{G_{p,q}}{T_{p,q}(f)}| = |\frac{G_{p,q}(f)}{T_{p,q}(f)}|\,|\frac{G_{p,q}}{G_{p,q}(f)}|$.

The theorem and the corollary are illustrated with examples in [8]. Maple worksheets verifying this and other results from [6–8] can be accessed from [10].

3 TRANSPOSITION SCALAR PRODUCT ON SPINOR SPACES

In [14, Ch. 18], Lounesto discusses scalar products on $S = C\ell_{p,q}f$ for simple Clifford algebras and on $\check{S} = S \oplus \hat{S} = C\ell_{p,q}e$, $e = f + \hat{f}$, for semisimple Clifford algebras where $\hat{f}$ denotes the grade involution of f. It is well known that in each case the spinor representation is faithful. Following Lounesto, we let $\check{\mathbb{K}}$ be either $\mathbb{K}$ or $\mathbb{K} \oplus \hat{\mathbb{K}}$ and $\check{S}$ be either S or $S \oplus \hat{S}$ when $C\ell_{p,q}$ is simple or semisimple, respectively. Then, in the simple algebras, the two β-scalar products are

$$S \times S \to \mathbb{K}, \quad (\psi, \phi) \mapsto \begin{cases} \beta_+(\psi,\phi) = s_1 \tilde{\psi}\phi \\ \beta_-(\psi,\phi) = s_2 \bar{\psi}\phi \end{cases} \tag{16}$$

whereas in the semisimple algebras they are

$$\check{S} \times \check{S} \to \check{\mathbb{K}}, \quad (\check{\psi}, \check{\phi}) \mapsto \begin{cases} (\beta_+(\psi,\phi), \beta_+(\psi_g,\phi_g)) = (s_1\tilde{\psi}\phi, s_1\tilde{\psi}_g\phi_g) \\ (\beta_-(\psi,\phi), \beta_-(\psi_g,\phi_g)) = (s_2\bar{\psi}\phi, s_2\bar{\psi}_g\phi_g) \end{cases} \tag{17}$$

for $\check{\psi} = \psi + \psi_g$ and $\check{\phi} = \phi + \phi_g$, $\psi, \phi \in S$, $\psi_g, \phi_g \in \hat{S}$, and where $\tilde{\psi}, \tilde{\psi}_g$ (resp. $\bar{\psi}, \bar{\psi}_g$) denotes reversion (resp. Clifford conjugation) of ψ, ψ_g. Here s_1, s_2 are special monomials in the Clifford algebra basis $\mathcal{B}$ which guarantee that the products $s_1\tilde{\psi}\phi$, $s_2\bar{\psi}\phi$, hence also $s_1\tilde{\psi}_g\phi_g$, $s_2\bar{\psi}_g\phi_g$, belong to $\mathbb{K} \cong \hat{\mathbb{K}}$.[9] In fact, the monomials s_1, s_2 belong to the chosen transversal of the stabilizer $G_{p,q}(f)$ in $G_{p,q}$ [8]. The automorphism groups of β_+ and β_- are defined in the simple case as, respectively, $G_+ = \{s \in C\ell_{p,q} \mid s\tilde{s} = 1\}$ and $G_- = \{s \in C\ell_{p,q} \mid s\bar{s} = 1\}$, and as ${}^2G_-$ and ${}^2G_+$ in the semisimple case. They are shown in [14, Tables 1 and 2, p. 236].

3.1 Simple Clifford algebras

In Example 3 [7] it was shown that the transposition scalar product in $S = C\ell_{2,2}f$ is different from each of the two Lounesto's products whereas Example 4 showed that the transposition product in $S = C\ell_{3,0}f$ coincided with β_+. Furthermore, it was remarked that $T_\varepsilon{}^{\tilde{}}(\psi)\phi$ always equaled β_+ for Euclidean signatures $(p, 0)$ and β_- for anti-Euclidean signatures $(0, q)$. We formalize this in the following proposition. For all proofs see [8].

Proposition 4. *Let $\psi, \phi \in S = C\ell_{p,q}f$ and $(\psi, \phi) \mapsto T_\varepsilon{}^{\tilde{}}(\psi)\phi = \lambda f$, $\lambda \in \mathbb{K}$, be the transposition scalar product. Let β_+ and β_- be the scalar products on S shown in (16). Then, there exist monomials s_1, s_2 in the transversal ℓ of $G_{p,q}(f)$ in $G_{p,q}$ such that*

$$T_\varepsilon{}^{\tilde{}}(\psi)\phi = \begin{cases} \beta_+(\psi,\phi) = s_1\tilde{\psi}\phi, & \forall \psi, \phi \in C\ell_{p,0}f, \\ \beta_-(\psi,\phi) = s_2\bar{\psi}\phi, & \forall \psi, \phi \in C\ell_{0,q}f. \end{cases} \tag{18}$$

[9] In simple Clifford algebras, the monomials s_1 and s_2 also satisfy: (i) $\tilde{f} = s_1 f s_1^{-1}$ and (ii) $\bar{f} = s_2 f s_2^{-1}$. The identity (i) (resp. (ii)) is also valid in the semisimple algebras provided $\beta_+ \not\equiv 0$ (resp. $\beta_- \not\equiv 0$).

Table 1 (Part 1): Automorphism group $G^{\varepsilon}_{p,q}$ of $T_{\varepsilon}\tilde{~}(\psi)\phi$

in simple Clifford algebras $C\ell_{p,q} \cong \mathrm{Mat}(2^k, \mathbb{R})$

$k = q - r_{q-p}$, $p - q \neq 1 \bmod 4$, $p - q = 0, 1, 2 \bmod 8$

(p,q)	$(0,0)$	$(1,1)$	$(2,0)$	$(2,2)$	$(3,1)$	$(3,3)$	$(0,6)$
$G^{\varepsilon}_{p,q}$	$O(1)$	$O(2)$	$\boxed{O(2)}$	$O(4)$	$O(4)$	$O(8)$	$\boxed{\boxed{O(8)}}$

Table 1 (Part 2): Automorphism group $G^{\varepsilon}_{p,q}$ of $T_{\varepsilon}\tilde{~}(\psi)\phi$

in simple Clifford algebras $C\ell_{p,q} \cong \mathrm{Mat}(2^k, \mathbb{R})$

$k = q - r_{q-p}$, $p - q \neq 1 \bmod 4$, $p - q = 0, 1, 2 \bmod 8$

(p,q)	$(4,2)$	$(5,3)$	$(1,7)$	$(0,8)$	$(4,4)$	$(8,0)$
$G^{\varepsilon}_{p,q}$	$O(8)$	$O(16)$	$O(16)$	$\boxed{\boxed{O(16)}}$	$O(16)$	$\boxed{O(16)}$

Let $u \in C\ell_{p,q}$ and let $[u]$ be a matrix of u in the spinor representation π_S of $C\ell_{p,q}$ realized in the spinor $(C\ell_{p,q}, \mathbb{K})$-bimodule ${}_{C\ell}S_{\mathbb{K}} \cong C\ell_{p,q}f\mathbb{K}$. Then, by [7, Prop. 5],

$$[T_{\varepsilon}\tilde{~}(u)] = \begin{cases} [u]^T & \text{if } p - q = 0, 1, 2 \bmod 8; \\ [u]^{\dagger} & \text{if } p - q = 3, 7 \bmod 8; \\ [u]^{\ddagger} & \text{if } p - q = 4, 5, 6 \bmod 8; \end{cases} \tag{19}$$

where T, $\dagger$, and $\ddagger$ denote, respectively, transposition, complex Hermitian conjugation, and quaternionic Hermitian conjugation. Thus, we immediately have:

Proposition 5. *Let $G^{\varepsilon}_{p,q} \subset C\ell_{p,q}$ where $C\ell_{p,q}$ is a simple Clifford algebra. Then, $G^{\varepsilon}_{p,q}$ is: The orthogonal group $O(N)$ when $\mathbb{K} \cong \mathbb{R}$; the complex unitary group $U(N)$ when $\mathbb{K} \cong \mathbb{C}$; or, the compact symplectic group $Sp(N) = U_{\mathbb{H}}(N)$ when $\mathbb{K} \cong \mathbb{H}$.[10] That is,*

$$G^{\varepsilon}_{p,q} = \begin{cases} O(N) & \text{if } p - q = 0, 1, 2 \bmod 8; \\ U(N) & \text{if } p - q = 3, 7 \bmod 8; \\ Sp(N) & \text{if } p - q = 4, 5, 6 \bmod 8; \end{cases} \tag{20}$$

where $N = 2^k$ and $k = q - r_{q-p}$.

The scalar product $T_{\varepsilon}\tilde{~}(\psi)\phi$ was computed with `CLIFFORD` [9] for all signatures (p,q), $p+q \leq 9$ [10]. Observe that as expected, in Euclidean (resp. anti-Euclidean) signatures $(p,0)$ (resp. $(0,q)$) the group $G^{\varepsilon}_{p,0}$ (resp. $G^{\varepsilon}_{0,q}$) coincides with the corresponding automorphism group of the scalar product β_+ (resp. β_-) listed in [14, Table 1, p. 236] (resp. [14, Table 2, p. 236]). This is indicated by a single (resp. double) box around the group symbol in Tables 1–5. For example, in Table 1, for the Euclidean signature $(2,0)$, we show $G^{\varepsilon}_{2,0}$ as $\boxed{O(2)}$ like for β_- whereas for the anti-Euclidean signature $(0,6)$, we show $G^{\varepsilon}_{0,6}$ as $\boxed{\boxed{O(8)}}$ like for β_+.

[10] See Fulton and Harris [12] for a definition of the quaternionic unitary group $U_{\mathbb{H}}(N)$. In our notation we follow *loc. cit.* page 100, 'Remark on Notations'.

Table 2 (Part 1): Automorphism group $G^{\varepsilon}_{p,q}$ of $T_{\varepsilon}\tilde{}(\psi)\phi$

in simple Clifford algebras $C\ell_{p,q} \cong \mathrm{Mat}(2^k, \mathbb{C})$

$k = q - r_{q-p}$, $p - q \neq 1 \bmod 4$, $p - q = 3, 7 \bmod 8$

(p,q)	$(0,1)$	$(1,2)$	$(3,0)$	$(2,3)$	$(0,5)$	$(4,1)$	$(1,6)$	$(7,0)$
$G^{\varepsilon}_{p,q}$	$U(1)$	$U(2)$	$\boxed{U(2)}$	$U(4)$	$\boxed{\boxed{U(4)}}$	$U(4)$	$U(8)$	$\boxed{U(8)}$

Table 2 (Part 2): Automorphism group $G^{\varepsilon}_{p,q}$ of $T_{\varepsilon}\tilde{}(\psi)\phi$

in simple Clifford algebras $C\ell_{p,q} \cong \mathrm{Mat}(2^k, \mathbb{C})$

$k = q - r_{q-p}$, $p - q \neq 1 \bmod 4$, $p - q = 3, 7 \bmod 8$

(p,q)	$(5,2)$	$(3,4)$	$(4,5)$	$(6,3)$	$(2,7)$	$(0,9)$	$(8,1)$
$G^{\varepsilon}_{p,q}$	$U(8)$	$U(8)$	$U(16)$	$U(16)$	$U(16)$	$\boxed{\boxed{U(16)}}$	$U(16)$

For simple Clifford algebras, the automorphism groups $G^{\varepsilon}_{p,q}$ are displayed in Tables 1, 2, and 3. In each case the form is positive definite and non-degenerate. Also, unlike in the case of the forms β_+ and β_-, there is no need for the extra monomial factor like s_1, s_2 in (16) (and (17)) to guarantee that the product $T_{\varepsilon}\tilde{}(\psi)\phi$ belongs to $\mathbb{K}$ since this is always the case [6, 7]. Recall that the only role of the monomials s_1 and s_2 is to permute entries of the spinors $\tilde{\psi}\phi$ and $\bar{\psi}\phi$ to assure that $\beta_+(\psi,\phi)$ and $\beta_-(\psi,\phi)$ belong to the (skew) field $\mathbb{K}$. That is, more precisely, that $\beta_+(\psi,\phi)$ and $\beta_-(\psi,\phi)$ have the form $\lambda f = f\lambda$ for some λ in $\mathbb{K}$. The idempotent f in the spinor basis in S corresponds uniquely to the identity coset $G_{p,q}(f)$ in the quotient group $G_{p,q}/G_{p,q}(f)$. Based on [7, Prop. 2] we know that since the vee group $G_{p,q}$ permutes entries of any spinor ψ, the monomials s_1 and s_2 belong to the transversal of the stabilizer $G_{p,q}(f) \lhd G_{p,q}$ [7, Cor. 2].[11]

One more difference between the scalar products β_+ and β_-, and the transposition product $T_{\varepsilon}\tilde{}(\psi)\phi$ is that in some signatures one of the former products may be identically zero whereas the transposition product is never identically zero. The signatures (p,q) in which one of the products β_+ or β_- is identically zero can be easily found in [14, Tables 1 and 2, p. 236] as the automorphism group of the product is then a general linear group.

3.2 Semisimple Clifford algebras

Faithful spinor representation of a semisimple Clifford algebra $C\ell_{p,q}$ ($p - q = 1 \bmod 4$) is realized in a left ideal $\check{S} = S \oplus \hat{S} = C\ell_{p,q}e$ where $e = f + \hat{f}$ for any primitive idempotent f. Recall that $\hat{}$ denotes the grade involution of $u \in C\ell_{p,q}$. We refer to [14, pp. 232–236] for some of the concepts. In particular, $S = C\ell_{p,q}f$ and $\hat{S} = C\ell_{p,q}\hat{f}$. Thus, every spinor $\check{\psi} \in \check{S}$ has unique components $\psi \in S$ and $\psi_g \in \hat{S}$. We refer to the elements $\check{\psi} \in \check{S}$ as "spinors" whereas to its components $\psi \in S$ and $\psi_g \in \hat{S}$ we refer as "$\frac{1}{2}$-spinors".

For the semisimple Clifford algebras $C\ell_{p,q}$, we will view spinors $\check{\psi} \in \check{S} = S \oplus \hat{S}$ as ordered pairs $(\psi, \psi_g) \in S \times \hat{S}$ when $\check{\psi} = \psi + \psi_g$. Likewise, we will view elements $\check{\lambda}$ in the double

[11] In [14, Page 233], Lounesto states correctly that "the element s can be chosen from the standard basis of $C\ell_{p,q}$." In fact, one can restrict the search for s to the transversal of the stabilizer $G_{p,q}(f)$ in $G_{p,q}$ which has a much smaller size $2^{q-r_{q-p}}$ compared to the size 2^{p+q} of the Clifford basis.

Table 3 (Part 1): Automorphism group $G^{\varepsilon}_{p,q}$ of $T_{\varepsilon}^{\tilde{}}(\psi)\phi$ in simple Clifford algebras $C\ell_{p,q} \cong \mathrm{Mat}(2^k, \mathbb{H})$

$k = q - r_{q-p}$, $p - q \neq 1 \bmod 4$, $p - q = 4, 5, 6 \bmod 8$

(p,q)	$(0,2)$	$(0,4)$	$(4,0)$	$(1,3)$	$(2,4)$	$(6,0)$
$G^{\varepsilon}_{p,q}$	$\boxed{\boxed{Sp(1)}}$	$\boxed{\boxed{Sp(2)}}$	$\boxed{Sp(2)}$	$Sp(2)$	$Sp(4)$	$\boxed{Sp(4)}$

Table 3 (Part 2): Automorphism group $G^{\varepsilon}_{p,q}$ of $T_{\varepsilon}^{\tilde{}}(\psi)\phi$ in simple Clifford algebras $C\ell_{p,q} \cong \mathrm{Mat}(2^k, \mathbb{H})$

$k = q - r_{q-p}$, $p - q \neq 1 \bmod 4$, $p - q = 4, 5, 6 \bmod 8$

(p,q)	$(1,5)$	$(5,1)$	$(6,2)$	$(7,1)$	$(2,6)$	$(3,5)$
$G^{\varepsilon}_{p,q}$	$Sp(4)$	$Sp(4)$	$Sp(8)$	$Sp(8)$	$Sp(8)$	$Sp(8)$

fields $\check{\mathbb{K}} = \mathbb{K} \oplus \hat{\mathbb{K}}$ as ordered pairs $(\lambda, \lambda_g) \in \mathbb{K} \times \hat{\mathbb{K}}$ when $\check{\lambda} = \lambda + \lambda_g$. As before, $\mathbb{K} = fC\ell_{p,q}f$ while $\hat{\mathbb{K}} = \hat{f}C\ell_{p,q}\hat{f}$. Recall that $\check{\mathbb{K}} \cong {}^2\mathbb{R} \stackrel{\text{def}}{=} \mathbb{R} \oplus \mathbb{R}$ or $\check{\mathbb{K}} \cong {}^2\mathbb{H} \stackrel{\text{def}}{=} \mathbb{H} \oplus \mathbb{H}$ when, respectively, $p - q = 1 \bmod 8$, or $p - q = 5 \bmod 8$.

In this section we classify automorphism groups of the transposition scalar product

$$\check{S} \times \check{S} \to \check{\mathbb{K}}, \quad (\check{\psi}, \check{\phi}) \mapsto T_{\varepsilon}^{\tilde{}}(\check{\psi}, \check{\phi}) \stackrel{\text{def}}{=} (T_{\varepsilon}^{\tilde{}}(\psi)\phi, T_{\varepsilon}^{\tilde{}}(\psi_g)\phi_g) \in \check{\mathbb{K}} \tag{21}$$

when $\check{\psi} = \psi + \psi_g$ and $\check{\phi} = \phi + \phi_g$.

Proposition 6. *Let $G^{\varepsilon}_{p,q} \subset C\ell_{p,q}$ where $C\ell_{p,q}$ is a semisimple Clifford algebra. Then, $G^{\varepsilon}_{p,q}$ is: The double orthogonal group ${}^2O(N) \stackrel{\text{def}}{=} O(N) \times O(N)$ when $\check{\mathbb{K}} \cong {}^2\mathbb{R}$ or the double compact symplectic group ${}^2Sp(N) \stackrel{\text{def}}{=} Sp(N) \times Sp(N)$ when $\check{\mathbb{K}} \cong {}^2\mathbb{H}$.[12] That is,*

$$G^{\varepsilon}_{p,q} = \begin{cases} {}^2O(N) = O(N) \times O(N) & \textit{when } p - q = 1 \bmod 8; \\ {}^2Sp(N) = Sp(N) \times Sp(N) & \textit{when } p - q = 5 \bmod 8; \end{cases} \tag{22}$$

where $N = 2^{k-1}$ and $k = q - r_{q-p}$.

The automorphism groups $G^{\varepsilon}_{p,q}$ for semisimple Clifford algebras $C\ell_{p,q}$ for $p + q \leq 9$ are shown in Tables 4 and 5. All results in these tables, like in Tables 1, 2, and 3, have been verified with `CLIFFORD` [9] and the corresponding Maple worksheets are posted at [10].

4 CONCLUSIONS

The transposition map $T_{\varepsilon}^{\tilde{}}$ allowed us to define a new transposition scalar product on spinor spaces. Only in the Euclidean and anti-Euclidean signatures, this scalar product is identical to the two known spinor scalar products β_+ and β_- which use, respectively, the reversion and the

[12] Recall that $Sp(N) = U_{\mathbb{H}}(N)$ where $U_{\mathbb{H}}(N)$ is the quaternionic unitary group [12].

Table 4: Automorphism group $G^{\varepsilon}_{p,q}$ of $T_{\varepsilon}^{\tilde{}}(\psi)\phi$ in semisimple Clifford algebras $C\ell_{p,q} \cong {}^2\mathrm{Mat}(2^{k-1}, \mathbb{R})$

$k = q - r_{q-p}$, $p - q = 1 \bmod 4$, $p - q = 1 \bmod 8$

(p,q)	$(1,0)$	$(2,1)$	$(3,2)$	$(0,7)$	$(4,3)$	$(1,8)$	$(5,4)$	$(9,0)$
$G^{\varepsilon}_{p,q}$	$\boxed{{}^2O(1)}$	${}^2O(2)$	${}^2O(4)$	$\boxed{\boxed{{}^2O(8)}}$	${}^2O(8)$	${}^2O(16)$	${}^2O(16)$	$\boxed{{}^2O(16)}$

Table 5: Automorphism group $G^{\varepsilon}_{p,q}$ of $T_{\varepsilon}^{\tilde{}}(\psi)\phi$ in semisimple Clifford algebras $C\ell_{p,q} \cong {}^2\mathrm{Mat}(2^{k-1}, \mathbb{H})$

$k = q - r_{q-p}$, $p - q = 5 \bmod 4$, $p - q = \bmod 8$

(p,q)	$(0,3)$	$(1,4)$	$(5,0)$	$(2,5)$	$(6,1)$	$(3,6)$	$(7,2)$
$G^{\varepsilon}_{p,q}$	$\boxed{\boxed{{}^2Sp(1)}}$	${}^2Sp(2)$	$\boxed{{}^2Sp(2)}$	${}^2Sp(4)$	${}^2Sp(4)$	${}^2Sp(8)$	${}^2Sp(8)$

conjugation and it is different in all other signatures. This new product is never identically zero and it does not require extra monomial factor to assure it is $\mathbb{K}$- or $\check{\mathbb{K}}$-valued. This is because the $T_{\varepsilon}^{\tilde{}}$ maps any spinor space to its dual. Then, we have identified the automorphism groups $G^{\varepsilon}_{p,q}$ of this new product in Tables 1–5 for $p + q = n \leq 9$. The classification is complete and sufficient due to the $\bmod 8$ periodicity.

We have observed the important role played by the idempotent group $T_{p,q}(f)$ and the field group $K_{p,q}(f)$ as normal subgroups in the stabilizer group $G_{p,q}(f)$ of the primitive idempotent f and their coset spaces $G_{p,q}/T_{p,q}(f)$, $G_{p,q}(f)/T_{p,q}(f)$, and $G_{p,q}/G_{p,q}(f)$ in relation to the spinor representation of $C\ell_{p,q}$. These subgroups allow to construct very effectively non-canonical transversals and hence basis elements of the spinor spaces and the (skew double) field underlying the spinor space. This approach to the spinor representation of $C\ell_{p,q}$ based on the stabilizer $G_{p,q}(f)$ of f leads to a realization that the Clifford algebras can be viewed as a twisted group ring $\mathbb{R}^t[(\mathbb{Z}_2)^n]$. In particular, we have observed that our transposition $T_{\varepsilon}^{\tilde{}}$ is then a 'star map' of $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ [17] which on a general twisted group ring $* : K^t[G] \to K^t[G]$ is defined as

$$\Big(\sum a_x \bar{x}\Big)^* = \sum a_x \bar{x}^{-1}.$$

This is because we recall properties of the transposition anti-involution $T_{\varepsilon}^{\tilde{}}$, and, in particular, its action $T_{\varepsilon}^{\tilde{}}(m) = m^{-1}$ on a monomial m in the Grassmann basis $\mathcal{B}$ which is, as we see now, identical to the action $*(m) = m^{-1}$ on every $m \in \mathcal{B}$. For a Hopf algebraic discussion of Clifford algebras as twisted group algebras, see [11, 15] and references therein.

REFERENCES

[1] R. Abłamowicz, *Clifford Algebra Computations with Maple*, in "Clifford (Geometric) Algebras with Applications in Physics, Mathematics, and Engineering", W. E. Baylis, (Ed.) (Birkhäuser, Boston, 1996) 463–502

[2] R. Abłamowicz, *Spinor Representations of Clifford algebras: A Symbolic Approach*, Computer Physics Communications Thematic Issue - Computer Algebra in Physics Research **115**, No. 2–3 (1998) 510–535

[3] R. Abłamowicz, *Computations with Clifford and Grassmann Algebras*. Adv. in Appl. Clifford Algebras, **19**, No. 3–4 (2009) 499–545

[4] R. Abłamowicz, *Computation of Non-Commutative Gröbner Bases in Grassmann and Clifford Algebras*. Adv. Applied Clifford Algebras 20 No. 3–4 (2010), 447–476

[5] R. Abłamowicz and B. Fauser, `GfG` - *Groebner for Grassmann - A Maple 12 Package for Groebner Bases in Grassmann Algebras*, http://math.tntech.edu/rafal/GfG12/ (2010)

[6] R. Abłamowicz and B. Fauser, *On the Transposition Anti-Involution in Real Clifford Algebras I: The Transposition Map*. Linear and Multilinear Algebra, Volume 59, Issue 12, December 2011, 1331–1358

[7] R. Abłamowicz and B. Fauser, *On the Transposition Anti-Involution in Real Clifford Algebras II: Stabilizer Groups of Primitive Idempotents*. Linear and Multilinear Algebra, Volume 59, Issue 12, December 2011, 1359–1381

[8] R. Abłamowicz and B. Fauser, *On the Transposition Anti-Involution in Real Clifford Algebras III: The Automorphism Group of the Transposition Scalar Product on Spinor Spaces* (to appear in Linear and Multilinear Algebra, DOI: 10.1080/03081087.2011.624093)

[9] R. Abłamowicz and B. Fauser, `CLIFFORD` with `Bigebra` – A Maple Package for Computations with Clifford and Grassmann Algebras, http://math.tntech.edu/rafal/ (©1996–2011)

[10] R. Abłamowicz and B. Fauser, Maple worksheets created with `CLIFFORD` for a verification of results presented in this paper and in [6, 7], http://math.tntech.edu/rafal/publications.html (©2011)

[11] H. Albuquerque and S. Majid, *Clifford Algebras Obtained by Twisting of Group Algebras*, J. of Pure and Appl. Algebra **171**, (2002) 133–148

[12] W. Fulton and J. Harris, *Representation Theory: A First Course*, (Springer, New York, 1991)

[13] A. J. Hahn, *Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups*, (Undergraduate Texts in Mathematics) (Springer-Verlag, New York, 1994)

[14] P. Lounesto, *Clifford Algebras and Spinors*, 2nd ed. (Cambridge University Press, Cambridge, 2001)

[15] S. Morier-Genoud and V. Ovsienko, *Simple graded commutative algebras*, J. of Algebra **323** (2010) 1649–1664

[16] Z. Oziewicz, *Clifford hopf-gebra and bi-universal hopf-gebra*, *Czechoslovak Journal of Physics* Volume 47, Number 12, 1267-1274, DOI: 10.1023/A:1022833801475

[17] D. S. Passman, *The Algebraic Structure of Group Rings*, (Robert E. Krieger Publishing Company, Malabar, Florida, 1985)

[18] I. R. Porteous, *Clifford Algebras and the Classical Groups*, (Cambridge University Press, Cambridge, 1995)

[19] J.J. Rotman, *Advanced Modern Algebra*, Revised Printing, (Prentice Hall, Upper Saddle River, 2002)

[20] N. Salingaros, *Realization, extension, and classification of certain physically important groups and algebras*, J. Math. Phys. **22** (1981) 226–232

[21] N. Salingaros, *On the classification of Clifford algebras and their relation to spinors in n dimensions*, J. Math. Phys. **23** (1) (1982) 1–7

[22] N. Salingaros, *The relationship between finite groups and Clifford algebras*, J. Math. Phys. **25** (1984) 738–742